



# Period determinant of an irregular connection over an elliptic curve

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## Abstract

In this article, we calculate the period of an irregular singular connection  $\nabla = d + dy$  on the Legendre curve  $U: y^2 = x(x-1)(x-\lambda)$ . We calculate its de Rham cohomology and the cycles in the homology of the dual connection and describe the period matrix. Terasoma's work is introduced to approximate the direct image connection  $\pi_*(\nabla)$  by a sequence of regular connections as an intermediate step where  $\pi: U \rightarrow \mathbb{A}^1 = \text{Spec } k[y]$ ,  $(x, y) \mapsto y$ . Finally, we will make the comparison of the period obtained by this approximation and that  $\nabla$  over  $U$ .

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## 1. Introduction

In this article, we will calculate the period determinant of the connection  $\nabla = d + dy$  over the Legendre elliptic curve:  $y^2 = x(x-1)(x-\lambda)$ .  $\nabla$  has an irregular singularity at  $\infty$ . The period for irregular connection is defined as a pairing of de Rham cohomology and a newly defined homology in [1]. It is shown to be perfect.

For a regular singular connection, the period is defined as a pairing of de Rham cohomology and the homology relative to the singularities valued in its dual local system, via a singular integration. From the view of the theory of characteristic classes, it was discussed

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profoundly by T. Saito and T. Terasoma in [2]. Especially, on  $\mathbb{P}^1$ , it is described explicitly as a product formula written in terms of the Gamma factors and the tame symbols by Terasoma in [4].

As for the irregular connections, very few cases are known. On  $\mathbb{A}^1 - \{\lambda_1, \dots, \lambda_n\}$ , when the connection is of the form  $d + \sum \frac{s_i}{x-\lambda_i} + dF$  for a polynomial  $F$  and positive real numbers  $s_i$ , the period determinant is calculated via approximation by series of rank 1 regular singular period also by Terasoma (cf. [5]).

In our case, as the forms are defined over an elliptic curve, the period determinant involves some elliptic integrals weighted by an exponential function, which are not easily calculated. The main idea is to calculate the period of its direct image connection under the second projection  $\pi: U \rightarrow \mathbb{A}^1$ ,  $(x, y) \mapsto y$ , in order to make it a higher rank connection on  $\mathbb{P}^1$ . Then the periods are determined to be integrations of differential forms over  $\mathbb{A}^1$  twisted by some monodromy and an exponential function, so that it can be approximated by a sequence of periods of properly chosen regular singular connections on  $\mathbb{P}^1$  converging to the irregular connection. Those periods of regular connections still do not satisfy some additional conditions for applying the product formula: specially those on the eigenvalues of the residues at the auxiliary regular singular points obtained while projecting down. Let  $D \in \mathbb{A}^1$  be those ramification points and  $d_D$  be the standard exterior differentiation of  $\mathcal{O}(-D)$  in  $\mathcal{O}$ . Tensoring with  $d_D$ ,  $\pi_*(\nabla)$  is approximated by connections whose periods are computable using the product formula.

Returning to the original connection, we take another connection  $\nabla_\Sigma = \nabla \otimes d_\Sigma$ , where  $\Sigma := \pi^{-1}D$ . This new connection on  $U$  produces the same periods as  $d_D \otimes \pi_*(\nabla)$  since they share an isomorphic relative homology group as well as an isomorphic de Rham cohomology. We will compare the period of  $\nabla$  with that of  $\nabla_\Sigma$  through the long exact sequences of de Rham cohomology and of the homology to produce exact value of the period determinant of  $\nabla$  on  $U$ . It will be treated in the generic case:  $\lambda^2 - \lambda + 1 \neq 0$ .

At the end, we calculate separately the period determinant at the special values of  $\lambda$ :  $\lambda^2 - \lambda + 1 = 0$ .

## 2. De Rham cohomology and homology for irregular singular connections

In this section, we calculate and describe the de Rham cohomology and the homology for  $\nabla = d + dy$  over an affine Legendre elliptic curve  $U: y^2 = x(x-1)(x-\lambda)$ . Let  $X$  be the completion of  $U$  in  $\mathbb{P}^2$  (i.e.,  $U = X - \{\infty\}$ ). Generally, for an integrable connection, the de Rham cohomology is defined as the hypercohomology of the de Rham complex.

Since  $\nabla$  admits no solution,  $H_{\text{dR}}^0(U, \nabla) = 0$  and

### Proposition 1.

$$H_{\text{dR}}^1(U, \nabla) = \frac{\Gamma(X, \Omega_X(5 \cdot \infty))}{\nabla \Gamma(X, \mathcal{O}_X(\infty))}.$$

Therefore,  $H_{\text{dR}}^1(U, \nabla)$  is generated by  $\frac{dx}{y}$ ,  $x \frac{dx}{y}$ ,  $dx$ ,  $x dx$ .

**Proof.** Since we take the cohomology on an affine variety, we have the identification

$$H_{\text{dR}}^1(U, \nabla) = \Gamma(X, \Omega_X(*\infty)) / \nabla \Gamma(X, \mathcal{O}_X(*\infty)).$$

It follows that the map  $\Gamma(X, \Omega((n+4) \cdot \infty)) \xrightarrow{\alpha} H_{\text{dR}}^1(U, \nabla)$  has the kernel  $\nabla \Gamma(X, \Omega(n \cdot \infty))$  and is surjective for  $n \gg 0$ . As

$$\dim \alpha \Gamma(X, \Omega((n+4) \cdot \infty)) = h^0(X, \Omega((n+4) \cdot \infty)) - h^0(X, \mathcal{O}(n \cdot \infty)) = 4$$

for  $n \geq 0$ , therefore, we have the identification. Since  $\frac{dx}{y}, x \frac{dx}{y}, dx, x dx, dy$  generates  $\Gamma(X, \Omega(4 \cdot \infty))$  and  $dy = \nabla(1), \frac{dx}{y}, x \frac{dx}{y}, dx, x dx$  is a basis for  $H_{\text{dR}}^1(U, \nabla)$ .  $\square$

We denote by  $H_*^{\text{irreg}}$  the homology for irregular connections defined by Bloch and Esnault in [1]. This homology is defined and studied over curves. It is known to make a perfect pair with the de Rham cohomology of the dual connection. The homology group has cycles that decays rapidly, as well as topological cycles.

From the duality, we see  $\dim H_0^{\text{irreg}}(U, \nabla^*) = 0$  and  $\dim H_1^{\text{irreg}}(U, \nabla^*) = 4$ , where  $\nabla^*$  is the dual connection of  $\nabla = d + dy$  over  $U$ .

Let  $\Delta$  be a small disk around  $\infty$ . Then the irregular homology groups are plugged into the following exact sequence:

$$\begin{aligned} 0 \rightarrow H_1(U, \mathbb{C}) &\xrightarrow{1 \otimes \exp y} H_1^{\text{irreg}}(U, \nabla^*) \rightarrow H_1^{\text{irreg}}(\Delta^*, \partial \Delta, \nabla^*) \\ &\xrightarrow{\delta} H_0(U, \mathbb{C}) \rightarrow H_0^{\text{irreg}}(U, \nabla^*) \rightarrow 0. \end{aligned} \quad (1)$$

In the above,  $H_1(U, \mathbb{C}) = H_1(X, \mathbb{C})$  is generated by  $\gamma_1, \gamma_2$ . The other cycles in  $H_1^{\text{irreg}}(U, \mathbb{C})$  are in the kernel of  $\delta$ . With the identification  $H_0(U, \mathbb{C}) = \mathbb{C}\langle \exp y \rangle$ , fixing a local parameter  $t = -(1/y)^{1/3}$  at  $\infty$ , we have the three rapid sectors where the solution decays rapidly:

$$\frac{2(i-1)\pi}{3} - \frac{\pi}{6} < \arg t < \frac{2(i-1)\pi}{3} + \frac{\pi}{6}, \quad \text{for } i = 1, 2, 3.$$

We call it the  $i$ th rapid decay sector. Let  $\eta_i$  be a chain from a fixed point  $p$  at  $\partial \Delta$  to  $\infty$  along the  $i$ th rapid decay sector.  $\eta_i \otimes \exp y$  for  $i = 1, 2, 3$  generate  $H_1^{\text{irreg}}(\Delta^*, \partial \Delta, \nabla^*)$ .  $\delta$  is the augmentation map  $\sum_i \alpha_i \eta_i \otimes \exp y \mapsto \sum_i \alpha_i \langle \exp y \rangle$ . Therefore the kernel is generated by  $(\eta_1 - \eta_2) \otimes \exp y$  and  $(\eta_2 - \eta_3) \otimes \exp y$ .

Therefore we obtain the following:

**Proposition 2.** Let  $\gamma_1, \gamma_2$  be the two generators of  $H_1(X, \mathbb{Z})$  and  $\gamma_3$  (respectively  $\gamma_4$ ) be the chain  $\eta_2 - \eta_1$  (respectively  $\eta_3 - \eta_2$ ).  $H_1^{\text{irreg}}(U, \nabla^*)$  has a basis  $\gamma_i \otimes \exp y$  for  $i = 1, 2, 3, 4$ .

Let  $w_i$  for  $i = 1, 2, 3, 4$  be a basis of  $H_{\text{dR}}^1(U, \nabla)$ . A period is the pairing of a cycle in  $H_1^{\text{irreg}}(U, \nabla^*)$  and a cocycle in  $H_{\text{dR}}^1(U, \nabla)$ , which is given by the integration

$$\langle \eta_i \otimes \exp y, \omega_j \rangle := \int_{\eta_i} \exp y \cdot \omega_j.$$

The period determinant is the determinant of the period matrix:

$$\text{per}(U, \nabla) := \det \left( \int_{\eta_i} \exp y \cdot \omega_j \right)_{i,j=1,2,3,4}.$$

Let  $k$  be the field of definition of  $U$ . It is not a well-defined number in  $\mathbb{C}$  since we can change the basis of the  $k$ -vector space  $H_{\text{dR}}^1(U, \nabla)$  by a matrix in  $\text{GL}(H_{\text{dR}}^1(U, \nabla)) = \text{GL}_4(k)$ . Moreover, we don't have a canonical choice of a basis in  $H_1^{\text{irreg}}(U, \nabla^*)$ . It depends on the choice of local solution of  $\nabla^*$ .

Throughout this article, we choose  $\exp y$  for the basis.

For the convenience in computations, a different basis for the cohomology as follows is used:

**Proposition 3.** Suppose  $\lambda \neq 0, 1$  and  $\lambda^2 - \lambda + 1 \neq 0$ . With the same notations as above,  $H_{\text{dR}}^1(U, \nabla)$  is generated by  $x dy$ ,  $xy dy$ ,  $xy^2 dy$ ,  $xy^3 dy$ .

**Proof.** It suffices to show nonvanishing of the determinant of the base change matrix from  $\mathcal{B} = \{\frac{dx}{y}, x \frac{dx}{y}, dx, x dx\}$  to  $\mathcal{B}' = \{xy^{i-1} dy \mid i \in [1, 4]\}$ . Equivalently,  $\det A \neq 0$ , where  $A$  is a  $(4 \times 4)$ -matrix such that  $\mathcal{B}' = A\mathcal{B}$ . For simplicity, denote  $xy^{i-1} dy$  by  $\omega'_i$ .

Firstly,  $\omega'_1 = x dy = \nabla(x) - dx \sim -\omega_3$ .

For  $i > 1$ ,  $\omega'_i = xy^{i-1} dy = \nabla(xy^{i-1}) - d(xy^{i-1}) = -ixy^{i-2} dy - y^{i-1} dx = -i\omega'_{i-1} - y^{i-1} dx$ . So we can replace  $\omega'_i$  with  $-y^{i-1} dx$  for  $i = 1, 2, 3, 4$ . Again, this can be replaced with  $y^{i-1} dx$  without changing the determinant. Abusing the notation, we mean  $y^{i-1} dy$  by  $\omega'_i$ . This does not change the determinant, either. Keeping this idea, we will calculate  $\tilde{\omega}'_i$  the class of  $\omega'_i$  in  $\mathcal{W}_i := \text{Span}\{\omega'_1, \dots, \omega'_i\} / \text{Span}\{\omega'_1, \dots, \omega'_{i-1}\}$ . If we replace  $\omega'_i$  with  $\tilde{\omega}'_i$ , this does not vary the determinant, since it is independent of extensions.

For simplicity in further calculations, we fix  $\gamma = -\frac{2}{9}(\lambda^2 - \lambda + 1)$  and  $\delta = \frac{t(t+1)}{9}$ , so that

$$\frac{y^2}{f'(x)} = \alpha x + \beta + \frac{\gamma x + \delta}{f'(x)} \quad (2)$$

for some  $\alpha, \beta$  which are polynomials in  $\lambda$ .  $\alpha$  and  $\beta$  won't be explicitly given here as they play no role in the evaluation of  $\det A$ . In the further calculations,  $\sim$  means that the two sides are in the same class of  $\mathcal{W}_i$ .

$$y dx = \frac{2y^2 dy}{f'(x)} = 2 \left( \alpha x + \beta + \frac{\gamma x + \delta}{f'(x)} \right) dy \sim (\gamma x + \delta) \frac{dx}{y} = \gamma \omega_2 + \delta \omega_1. \quad (3)$$

Similarly,

$$\begin{aligned} y^2 dx &= y^2 \frac{2y dy}{f'(x)} = 2y \left( \alpha x + \beta + \frac{\gamma x + \delta}{f'(x)} \right) dy \\ &\sim (\gamma x + \delta) dx = \gamma \omega_4. \end{aligned} \quad (4)$$

Finally,

$$\begin{aligned} y^3 dx &= 2y^2 \left( \alpha x + \beta + \frac{\gamma x + \delta}{f'(x)} \right) dy \\ &\sim y(\gamma x + \delta) dx \sim \gamma xy dx = 2\gamma x \frac{y^2}{f'(x)} dy \\ &= 2\gamma x \left( \alpha x + \beta + \frac{\gamma x + \delta}{f'(x)} \right) dy \sim 2\alpha \gamma x^2 dy - \gamma x(\gamma x + \delta) \frac{dx}{y} \\ &= 2\alpha \gamma (\nabla(x^2) - 2x dx) + \gamma^2 x^2 \frac{dx}{y} + \gamma \delta x \frac{dx}{y} = \gamma \delta \omega_2 + 2\gamma^2 \frac{x^2 dy}{f'(x)} \\ &= \gamma \delta \omega_2 + 2\gamma^2 \left( \frac{1}{3} + \frac{2(\lambda + 1)x - \lambda}{3f'(x)} \right) dy \sim \gamma \delta \omega_2 + \gamma^2 \left( \frac{2(\lambda + 1)x - \lambda}{3} \right) \frac{dx}{y} \\ &= -\frac{\gamma^2 \lambda}{3} \omega_1 + \gamma \left( \delta + \frac{2}{3}(\lambda + 1)\gamma \right) \omega_2. \end{aligned} \quad (5)$$

Therefore,

$$\begin{aligned} \det A &= \begin{vmatrix} 0 & 0 & 1 & 0 \\ \delta & \gamma & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ -\frac{\gamma^2 \lambda}{3} & \gamma(\delta + \frac{2}{3}\gamma(\lambda + 1)) & 0 & 0 \end{vmatrix} \\ &= -\gamma^2 \left( \delta^2 + \frac{2}{3}\delta\gamma(\lambda + 1) + \frac{\gamma^2 \lambda}{3} \right) \\ &= \frac{2^2}{3^7} \lambda^2 (\lambda - 1)^2 (\lambda^2 - \lambda + 1)^2. \end{aligned} \quad (6)$$

This never vanishes on the domain of  $\lambda$  in consideration.  $\square$

With respect to this basis of cohomology, the period determinant is calculated as a well-defined complex number. As this is varied only by a basis change in  $\mathrm{GL}_4(k)$ , the period determinant is in the same class of complex numbers  $\mathbb{C}^*/k^*$  as the one calculated using the basis of Proposition 1. More precisely, the period determinant for  $\omega'_i$  is  $\det A$  in the above times the period determinant for  $\omega_i$ . This comparison of period determinant is valid for the Legendre elliptic curves when  $\lambda^2 - \lambda + 1 \neq 0$ . This is why the exceptional case is separately treated at the end.

### 3. Direct image connection

We will keep the same convention as in the previous section.

In the previous section, we have expressed the period as an exponential elliptic integration. As it is known, elliptic integrations are difficult to evaluate, thus we take its direct image  $\pi_*(\nabla)$  onto  $\mathbb{A}^1$  where  $\pi : (x, y) \mapsto y$  for  $(x, y) \in U$ . Note  $\nabla = \pi^*(d + dy)$ . Using the projection formula, we have

$$\begin{aligned}\pi_*(\mathcal{O}_U, \nabla) &= \pi_*\pi^*(\mathcal{O}_{\mathbb{A}^1}, d + dy) \\ &= (\mathcal{O}_{\mathbb{A}^1}, d + dy) \otimes \pi_*(\mathcal{O}_U, d).\end{aligned}\quad (7)$$

Let  $f(x)$  be  $x(x-1)(x-\lambda)$  throughout this article and  $L$  be the function field of the elliptic curve  $k(x, y)/(y^2 - f(x))$ . The trace map of  $L$  in  $k(y)$  induces a map

$$\pi_*(\mathcal{O}_U) \xrightarrow{\text{Tr}_{L/k(y)}} \mathcal{O}_{\mathbb{A}^1}.$$

Since this map has a section and flat, we have splitting of  $(\pi_*(\mathcal{O}_U), d)$  as connection:  $(\pi_*(\mathcal{O}_U), d) = (\mathcal{O}_{\mathbb{A}^1}, d) \oplus (V, \nabla')$  for a rank 2 connection on the kernel of  $\text{Tr}_{L/k(y)}$ , which is generated by

$$e_1 := -\frac{\lambda+1}{3} + x \quad \text{and} \quad e_2 := -\frac{\lambda^2+1}{3} + x^2.$$

Therefore we have

$$\pi_*(\mathcal{O}_U, \nabla) = (\mathcal{O}_{\mathbb{A}^1}, d + dy) \oplus ((\mathcal{O}_{\mathbb{A}^1}, d + dy) \otimes (V, \nabla')).$$

Let  $x_1, x_2$  be the roots of  $f'(x)$ . Note that  $\pi_*(\mathcal{O}_U, \nabla)$  obtains regular singularities via ramifications at the points  $(x_i, \pm\sqrt{f(x_i)})$  for  $i = 1, 2$ , which are concentrated in  $(V, \nabla')$ .

With respect to the basis  $\{e_1, e_2\}$ , the connection matrix of  $\nabla'$  is

$$\frac{2y dy}{3(y^2 - f(x_1))(y^2 - f(x_2))} \cdot \begin{pmatrix} \frac{1}{9}(\lambda+1)(2\lambda^2 - 3\lambda + 2) + y^2 & \frac{2}{9}\lambda(\lambda+1) - \frac{2}{3}(\lambda+1)y^2 \\ -\frac{2}{9}(\lambda^2 - \lambda + 1) & -\frac{2}{9}\lambda(\lambda+1) + 2y^2 \end{pmatrix}$$

which has simple poles at the zeroes of  $y = \pm\sqrt{f(x_i)}$  for  $i = 1, 2$ .

We have two different configurations of singularities after  $\lambda$ : one is the generic case when  $\lambda^2 - \lambda + 1 \neq 0$  and the other is when  $\lambda^2 - \lambda + 1 = 0$ . In the first case, we have four different singular points at  $y = \pm\sqrt{f(x_1)}, \pm\sqrt{f(x_2)}$ . In the second case, we have only two singular points  $y = \pm\sqrt{f(x_1)}$ .

In either case, we will approximate the answer by a sequence of regular singular connections over  $\mathbb{P}^1$ .

#### 4. Product formula: Terasoma's work

In this section, we recall the main theorem in [4]. The theorem tells the exact value of the period determinant of a regular singular connection on  $\mathbb{A}^1$  for a canonical choice of basis of de Rham cohomology and relative homology valued in the dual local system, assuming some extra conditions. This result will be applied for the approximation later.

We will firstly recall some necessary notions for the result.

Let  $D = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  be  $n$  distinct points in  $\mathbb{A}^1$ . A logarithmic connection with poles at  $D \cup \{\infty\}$

$$\nabla : \mathcal{V} \rightarrow \mathcal{V} \otimes \Omega_{\mathbb{P}^1}(\log(D + \infty))$$

on a trivial vector bundle  $\mathcal{V}$  of rank  $r$  can be written as

$$\nabla = d + \sum_{i=1}^n \frac{B^{(i)}}{x - \lambda_i} dx,$$

where  $B^{(i)} = \text{Res}_{\lambda_i}(\nabla) \in \text{End}(\mathbb{C}^r)$  is the residue of  $\nabla$  at  $\lambda_i$ . And the residue at  $\infty$  is  $B^{(\infty)} := \sum_{i=1}^n (-B^{(i)})$ . Throughout this section, we assume that no two eigenvalues of the residue are different by an integer and that they have positive real part.

**Definition 4.** Let  $(V, \nabla)$  be a connection on  $\mathbb{A}^1 - D$ .  $(\tilde{V}, \tilde{\nabla})$  be its logarithmic extension to  $\mathbb{P}^1$ . If no eigenvalue of the residue of  $\nabla$  at  $D$  is a nonpositive integer, it is called a small extension of  $\nabla$  along  $D$ .

When  $\nabla$  is small at  $D$ , then the de Rham cohomology  $H_{\text{dR}}^1(\mathbb{A}^1, \nabla)$  of the logarithmic de Rham complex is generated by

$$\left( \frac{1}{x - \lambda_i} - \frac{1}{x - \lambda_{i+1}} \right) dx \otimes w, \quad \text{for } i = 1, 2, \dots, n-1,$$

where  $w$  is a vector in  $V$ . Thus  $\text{rank } H_{\text{dR}}^1(\mathbb{A}^1, \nabla) = r(n-1)$ , which is canonically isomorphic to  $H_{\text{dR}}^1(\mathbb{A}^1, j_* j^*(\nabla))$ , where  $j : \mathbb{A}^1 - D \hookrightarrow \mathbb{A}^1$  is the open embedding (cf. [2,4]).

Now we need the corresponding homology theory to yield a perfect pairing with the previously described de Rham cohomology. This is the relative homology of the pair  $(\mathbb{A}^1, D)$  valued in the local system of  $\nabla^*$ . It is isomorphic to the relative homology of the pair  $(U, \bigcup_i D_i)$  valued in the dual local system, where  $D_i$  is a sufficiently small disk around  $\lambda_i$ . It can be seen as Borel–Moore homology valued in the local system. A cycle in this homology group can be written as linear sum of  $\gamma_\alpha \otimes f_\alpha$ , where  $\gamma_\alpha$  is a cycle in  $H_i(\mathbb{A}^1, D, \mathbb{Z})$  and  $f_\alpha$  is a branch of a solution of  $\nabla^*$  over  $\gamma_\alpha$ . The main theorem tells us that  $H_1(\mathbb{A}^1, D, \nabla^*)$  and  $H_{\text{dR}}^1(\mathbb{A}^1, \nabla)$  make a perfect pair via integration.

For a form  $\omega := \left(\frac{1}{x-\lambda_i} - \frac{1}{x-\lambda_{i+1}}\right) dx \otimes v$  in  $H_{\text{dR}}^1(\mathbb{A}^1, \nabla)$ , and  $\delta := \gamma \otimes f$  in  $H_1(\mathbb{A}^1, D, \nabla^*)$ , we define their pairing as

$$\langle \omega, \delta \rangle := \int_{\gamma} \left( \frac{1}{x-\lambda_i} - \frac{1}{x-\lambda_{i+1}} \right) \langle v, f \rangle dx.$$

The product formula à la Terasoma shows that the above pairing is perfect. To state the formula we need to introduce the tame symbol and the Gamma factor of a connection.

For a rank 1 connection  $\nabla = d + \sum_{i=1}^n \frac{b^i}{x-\lambda_i}$  over  $\mathbb{A}^1$  with log poles at  $\lambda_1, \dots, \lambda_n$ , where  $b^i \in \mathbb{C}$  is the residue at  $\lambda_i$ . Assume, as before,  $b_i$  have positive real part. Let  $p$  be a fixed point in  $\mathbb{A}^1 - \{\lambda_1, \dots, \lambda_n\}$  and  $\gamma_i$  be a fixed path from  $p$  to  $\lambda_i$ .  $\nabla^*$  has a multi-valued solution  $\prod_{i=1}^n (x-\lambda_i)^{b^i}$ . Fix a branch of  $\log(x-\lambda_i)$  around  $\lambda_i$  to have real value at  $\lambda_i + \epsilon$  for small  $\epsilon \in \mathbb{R}_{>0}$ , thus we have fixed the branch of  $(x-\lambda_i)^{b^i} = \exp(b^i \log(x-\lambda_i))$  as well. Let  $D(x)_{\gamma_i}$  be the branch of the above solution of  $\nabla^*$  on  $\gamma_i$  according to the chosen branch of the logarithm.

The *tame symbol* of the rank 1 connection  $\nabla$  is a value depending on the path from  $p$  to  $\lambda_i$

$$(\nabla, (x-\lambda_i))_{\gamma_i} := \lim_{\substack{x \rightarrow \lambda_i \\ \text{along } \gamma_i}} \frac{D(x)_{\gamma_i}}{(x-\lambda_i)^{b^i}} = \prod_{j \neq i} (\lambda_i - \lambda_j)^{b^j}$$

and for a path  $\gamma_{\infty}$  from  $p$  to  $\infty$ ,

$$\left( \nabla, \frac{1}{x} \right)_{\gamma_{\infty}} := \lim_{x \rightarrow \infty} D(x)_{\gamma_{\infty}} x^{b^{\infty}}.$$

In general, for a higher rank connection, we define the *tame symbol* as that of the determinant connection.

Keeping the assumption that the residue at  $\lambda_i$  has only positive real part, we define the Gamma factor as a generalization of the Gamma function for the residue:

$$\Gamma_{\lambda}(\nabla) = \begin{cases} \det \left( \int_0^{\infty} x^{\text{Res}_{\lambda}(\nabla)} e^{-x \frac{dx}{x}} \right) & \text{for } \lambda \in D, \\ \det \left( \int_0^{\infty} x^{-\text{Res}_{\infty}(\nabla)} e^{-x \frac{dx}{x}} \right) & \text{for } \lambda = \infty. \end{cases}$$

Now, let  $\nabla$  be a connection with log poles at  $D = \{\lambda_1, \dots, \lambda_n\}$  and  $\infty$ . Assume the eigenvalues of  $\text{Res}_{\lambda_i}(\nabla)$  have positive real parts and they are not different by an integer. Let  $\{e_i\}_{i=1, \dots, r}$  be a basis for the underlying vector space where the connection values and  $\{e_j^*\}$  be the dual basis. Let  $\delta_j := \gamma_j - \gamma_{j-1}$  and  $\text{Sol}(\nabla^*)(e_q^*)$  be the branch of  $\text{Sol}(\nabla^*)$  analytically continued along  $\gamma_i$  with the initial value  $e_q^*$  at  $p$ . We take basis

$$\omega_i(e_p) := \left( \frac{1}{x-\lambda_i} - \frac{1}{x-\lambda_{i+1}} \right) dx \otimes e_p \quad \text{in } H_{\text{dR}}^1(\mathbb{A}^1, \nabla),$$



and

$$\delta_j(e_q^*) := \delta_j \otimes \text{Sol}(\nabla^*)(e_q^*),$$

to define a period matrix for  $(i, j)$  as

$$A_{ij} := (\langle \delta_j(e_q^*), \omega_i(e_p) \rangle)_{1 \leq p, q \leq r}.$$

**Theorem 5** (Terasoma [4]). *The determinant of the period matrix with respect to the above basis of  $H_{\text{dR}}^1(\mathbb{A}^1, \nabla)$  and  $H_1(\mathbb{A}^1, D, \nabla^*)$  is*

$$\det(A_{ij})_{1 \leq i, j \leq n-1} = \prod_{i=1}^n (\nabla, x - \lambda_i)_{\gamma_i} \cdot \left( \nabla, \frac{1}{x} \right)^{-1} \prod_{\gamma \in \infty}^n \Gamma_{\lambda_i}(\nabla) \cdot \Gamma_{\infty}(\nabla)^{-1}$$

in  $\mathbb{C}^*$ . Therefore  $H_{\text{dR}}^1(\mathbb{A}^1, \nabla)$  and  $H_1(\mathbb{A}^1, D, \nabla^*)$  make a perfect pairing because tame symbols and Gamma factors never vanish.

**Proof.** See [4].  $\square$

We can take a different basis for  $H_{\text{dR}}^1(\mathbb{A}^1, \nabla)$  to evaluate the period determinant:

$$\eta_i(e_p) := \frac{x^{i-1}}{\prod_{k=1}^n (x - \lambda_k)} \otimes e_p, \quad \text{for } i = 1, \dots, n-1 \text{ and } p = 1, \dots, r.$$

**Proposition 6.** *The period determinant of the pairing of  $H_{\text{dR}}^1(\mathbb{A}^1, \nabla)$  and  $H_1(\mathbb{A}^1, D, \nabla^*)$  with respect to the basis  $\eta_i(e_p)$  and  $\delta_j(e_q^*)$  is*

$$\det \left( \int_{\delta_j(e_q^*)} (\eta_i(e_p)) \right) = \frac{\det \left( \int_{\delta_j(e_q^*)} \omega(e_p) \right)}{\Delta(\lambda_1, \dots, \lambda_n)^r}$$

in  $\mathbb{C}^*$  where  $\Delta(\lambda_1, \dots, \lambda_n) = \prod_{i < j} (\lambda_i - \lambda_j)$  the Vandermonde determinant of  $(\lambda_1, \dots, \lambda_n)$ .

**Proof.** We will prove it for rank 1 case. It will be directly generalize for higher rank cases. Set

$$\eta'_i = \prod_{k=1}^{i-1} (x - \lambda_k) \eta_i \quad \text{for } \eta_i = \frac{x^{i-1} dx}{\prod_{k=1}^n (x - \lambda_k)}.$$

Then we have  $x\eta'_i = \eta'_{i+1} + \lambda_i \eta_i$ . By induction on  $i$ , it follows that  $\eta_i = \eta'_i + \sum_{k>i} c_k \eta'_k$  for some constants  $c_k$ . Hence

$$\det \left( \int_{\delta_j} \eta_i \right) = \det \left( \int_{\delta_j} \eta'_i \right).$$

Whereas, for  $a_i = \text{Res}_{\lambda_i} \eta'_i$ ,

$$\eta'_i = \sum_{k=i}^n \frac{a_k dx}{x - \lambda_k} = \sum_{k=i}^{n-1} \sum_{\ell=i}^k a_\ell \omega_k + \sum_{\ell=i}^n \frac{a_\ell dx}{x - \lambda_n}.$$

Since  $\text{Res}_\infty \eta'_i = 0$ ,  $\sum_{k=i}^n a_k$  the sum of all residues of  $\eta'$  is 0.

$a_i = \text{Res}_{\lambda_i} \eta'_i = \frac{1}{\prod_{k=i}^n (\lambda_i - \lambda_k)}$  and we have

$$\eta'_i = \frac{\omega_i}{\prod_{k \geq i} (\lambda_i - \lambda_k)} + \sum_{k > i} c_k \omega_k$$

for some constants  $c_k$ .

Therefore the period determinant is

$$\det \left( \int_{\delta_j} \eta_i \right) = \frac{1}{\prod_{i < j} (\lambda_i - \lambda_j)} \det \left( \int_{\delta_j} \omega_i \right),$$

which finishes the proof for rank 1 case. In higher rank case, we have multiple contribution of the constant to its rank. Hence we obtain the power of the Vandermonde matrix in the denominator.  $\square$

In [3] and [6], the authors treat the determinant of the period matrix

$$\left( \int_{\delta_j} \prod_{k=1}^n (x - \lambda_k)^{s_k-1} x^{i-1} dx \right)_{1 \leq i, j \leq n-1},$$

where  $\delta_i$  is as above. The integration of each entry is, in fact, the pairing

$$\int_{\delta_j} \prod_{k=1}^n (x - \lambda_k)^{s_k-1} x^{i-1} dx = \left\langle \delta_j \otimes \prod_{k=1}^n (x - \lambda_k)^{s_k}, \eta_i \right\rangle$$

for a rank 1 connection  $\nabla = d + \sum_{k=1}^n \frac{s_k}{x - \lambda_k} dx$  on  $\mathbb{A}^1$  and  $\eta_i$  are the same as we have described before.

Using Proposition 6, we obtain the period determinant of the above connection as

$$\det \left( \int_{\delta_j} \prod_{k=1}^n (x - \lambda_k)^{s_k-1} x^{i-1} dx \right) = \frac{\Gamma(s_1) \cdots \Gamma(s_n)}{\Gamma(s_1 + \cdots + s_n)} \prod_{i < j} (\lambda_j - \lambda_i) \prod_{i=1}^n \prod_{j \neq i} (\lambda_j - \lambda_i)^{s_i-1}$$

in  $\mathbb{C}^*$ .

## 5. Period integral

In this section, we want to evaluate the integration via an approximation. As we have seen earlier,  $\pi_*(\nabla)$  splits into  $(\mathcal{O}_{\mathbb{A}^1}, d + dy) \oplus (V, \nabla')$  for a rank 2 connection  $\nabla'$ . Firstly, before applying the results of Terasoma for the approximation, we have to check if the rank 2 part of  $\pi_*(\nabla)$  satisfies the condition on eigenvalues of the residues. Let  $D = \{z_1, \dots, z_n\}$  be a nonempty set of points in  $\mathbb{A}^1$ . For a connection  $\nabla$  over  $\mathbb{A}^1$ , possibly singular, we denote by  $d_D$  the standard exterior differentiation on  $\mathcal{O}_{\mathbb{A}^1}(-D)$  and define  $\nabla_D := \nabla \otimes d_D$ .

When  $\lambda^2 - \lambda + 1 \neq 0$ , the residue of  $\nabla'$  at  $\{\pm\sqrt{f(x_i)}\}$  has two eigenvalues 0 and  $-1/2$ . This means Terasoma's result is not directly applicable. In case  $\lambda^2 - \lambda + 1 = 0$ , the eigenvalues are  $1/3, 2/3$ .

For the generic case  $\lambda^2 - \lambda + 1 \neq 0$ , we modify  $\nabla$  to satisfy the eigenvalue condition, tensoring the rank 1 connection  $(\mathcal{O}_{\mathbb{A}^1}(-D), d + \varpi)$  where

$$\varpi = \left( \frac{1}{y - \sqrt{f(x_1)}} + \frac{1}{y + \sqrt{f(x_1)}} + \frac{1}{y - \sqrt{f(x_2)}} + \frac{1}{y + \sqrt{f(x_2)}} \right) dy.$$

Then the new connection obtained is  $(\mathcal{O}_{\mathbb{A}^1}(-D), d + \varpi) \otimes \pi_*(\mathcal{O}_U, \nabla) = \pi_*\pi^*(\mathcal{O}_{\mathbb{A}^1}(-D), d + dy + \varpi)$  and this has regular singularities at  $D$  with the eigenvalues 1,  $3/2$  of the residues at  $D$ .

We will calculate the period determinant for the above connection  $\pi_*(\nabla_D)$ . The generic case  $\lambda^2 - \lambda + 1 \neq 0$  will be treated first. Since  $\pi_*(\nabla_D)$  splits into a direct sum of a rank 1 connection and a rank 2 connection, the period determinant is the product of the period of rank 1 part and that of rank 2 part. The period determinant of a rank 1 irregular connection of the form  $\nabla = d + dF + \sum_{i=1}^n \frac{s_i dx}{x - \lambda_i}$  was evaluated by Terasoma:

**Theorem 7** (Terasoma [5]). *Let  $\nabla = d + dF + \sum_{i=1}^n \frac{s_i dx}{x - \lambda_i}$  be a connection on  $\mathbb{A}^1$  for a polynomial  $F$ . Let  $I_i$  be a fixed path from  $\lambda_i$  to  $\lambda_{i+1}$  for  $i = 1, \dots, n-1$  and  $J_i$  be a fixed path from  $\lambda_n$  to  $\infty$  along the  $i$ th rapid decay sector for  $i = 1, \dots, \deg(F)$ . Taking the basis  $\eta_j = \frac{x^{j-1} dx}{\prod_{i=1}^n (x - \lambda_i)}$  of  $H_{\text{dR}}^1(\mathbb{A}^1, \nabla)$  and the basis  $\delta_j := I_j \otimes \prod_{k=1}^n (x - \lambda_k)^{s_k} \exp(F(x))$  ( $j = 1, \dots, n-1$ ) and  $J_j \otimes \prod_{k=1}^n (x - \lambda_k)^{s_k} \exp(F(x))$  ( $j = 1, \dots, \deg(F)$ ), we have the determinant of the period matrix*

$$\begin{aligned} D &= \det(\langle \delta_j, \eta_i \rangle) \\ &= (2\pi)^{(d-1)/2} \Gamma(s_1) \cdots \Gamma(s_n) (da_d)^{-s-(d-1)/2} (-1)^{ds+d(d-1)/4} \\ &\quad \times \prod_{i=1}^n \left( \prod_{j \neq i} (\lambda_i - \lambda_j)^{s_i-1} \prod_{i < j} (\lambda_j - \lambda_i) \right) \\ &\quad \times \prod_{i=1}^n \exp(F(\lambda_i)) \prod_{F'(u)=0} \exp(F(u)), \end{aligned} \tag{8}$$

in  $\mathbb{C}^*$ , where  $s = s_1 + \cdots + s_n$ .

The rank 1 part of  $\pi_*(\nabla_D)$  is  $d + dy + \varpi$ . Applying the formula, we have the period determinant of the rank 1 part

$$\begin{aligned} D &= (\sqrt{f(x_1)} + \sqrt{f(x_2)})(2\sqrt{f(x_1)}2\sqrt{f(x_2)})(\sqrt{f(x_1)} - \sqrt{f(x_2)}) \\ &\quad \times (\sqrt{f(x_1)} - \sqrt{f(x_2)})(-2\sqrt{f(x_2)})(-\sqrt{f(x_2)} - \sqrt{f(x_1)}) \\ &= 2^2 \sqrt{f(x_1)} \sqrt{f(x_2)} (f(x_1) - f(x_2))^2 \end{aligned} \quad (9)$$

in  $\mathbb{C}^*$ .

For simplicity, we will denote

$$\omega = \frac{2y \, dy}{3(y^2 - f(x_1))(y^2 - f(x_2))} \in \Gamma(\mathbb{A}^1, \Omega(\log D)).$$

The rank 2 part has the connection

$$\begin{aligned} \nabla_1 &= (d + dy) \otimes \nabla'_D \\ &= d + Idy + \varpi \\ &\quad + \begin{pmatrix} \frac{1}{9}(\lambda + 1)(2\lambda^2 - 3\lambda + 2) + y^2 & \frac{2}{9}\lambda(\lambda^2 + 1) - \frac{2}{3}(\lambda + 1)y^2 \\ -\frac{2}{9}(\lambda^2 - \lambda + 1) & -\frac{2}{9}\lambda(\lambda + 1) + 2y^2 \end{pmatrix} \omega. \end{aligned} \quad (10)$$

We are now prepared to approximate the integrations. Note that the residue of the new rank 2 connection has 1, 3/2 at each point of  $D$ .

Let  $e_1, e_2$  be the standard basis of  $\Gamma(\mathbb{P}^1, \mathcal{O}^{\oplus 2})$  and  $e_1^*, e_2^*$  be the dual so that  $\langle e_i, e_j^* \rangle = \delta_{ij}$ . Let  $\gamma_i$  be a fixed path from 0 to  $-\sqrt{f(x_1)}$ ,  $\sqrt{f(x_1)}$ ,  $-\sqrt{f(x_2)}$ ,  $\sqrt{f(x_2)}$  respectively, for  $i = 1, 2, 3, 4$ .  $\gamma_\infty$  is a path from 0 to  $\infty$  along the rapid decaying sector around  $\infty$ .  $I_j$  is the chain  $\gamma_{i+1} - \gamma_i$  for  $i = 1, 2, 3$  and  $I_4 := \gamma_\infty - \gamma_4$ . With this notation, we have the following cycles as a basis for  $H_1^{\text{irreg}}(\mathbb{A}^1, D, \nabla_1^*)$ :

$$C_{j,b} := I_j \otimes (\exp y \cdot \text{Sol}(\nabla_1^*)) (e_b^*), \quad \text{for } j = 1, \dots, 4 \text{ and } b = 1, 2,$$

and  $H_{\text{dR}}^1(\mathbb{A}^1, \nabla_1)$  has a basis

$$\eta_{i,a} := \frac{y^{i-1} dy}{(y - \sqrt{f(x_1)})(y + \sqrt{f(x_1)})(y - \sqrt{f(x_2)})(y + \sqrt{f(x_2)})} \otimes e_a \quad (11)$$

for  $i = 1, 2, 3, 4$  and  $a = 1, 2$ .

The pairing of  $\eta_{i,a}$  and  $C_{j,b}$  is the integration

$$\begin{aligned} P_{i,j,a,b} &:= \langle \eta_{i,a}, C_{j,b} \rangle \\ &= \int_{I_j} \frac{y^{i-1} \exp y \langle e_a, \text{Sol}(\nabla_1^*)(e_b^*) \rangle dy}{(y - \sqrt{f(x_1)})(y + \sqrt{f(x_1)})(y - \sqrt{f(x_2)})(y + \sqrt{f(x_2)})}. \end{aligned} \quad (12)$$

Let  $I_j^{(m)}$  be  $I_j$  for  $i = 1, 2, 3$  and  $I_4^{(m)}$  be  $\gamma_{(-m)} - \gamma_4$ , where  $\gamma_{(-m)}$  is a path from 0 to  $-m$  sitting in the rapid decay sector of  $\nabla_1^*$  near  $\infty$  and converges uniformly to  $\gamma_\infty$ . The above integration is approximated by

$$\begin{aligned} P_{(m),i,j,a,b} &= \int_{I_j^{(m)}} \frac{y^{i-1} \left(1 + \frac{y}{m}\right)^m \langle e_a, \text{Sol}(\nabla_1^*)(e_b^*) \rangle dy}{(y - \sqrt{f(x_1)})(y + \sqrt{f(x_1)})(y - \sqrt{f(x_2)})(y + \sqrt{f(x_2)})} \\ &= \left(\frac{1}{m}\right)^m \int_{I_j^{(m)}} \frac{y^{i-1} (y+m)^m \langle e_a, \text{Sol}(\nabla_1^*)(e_b^*) \rangle dy}{(y - \sqrt{f(x_1)})(y + \sqrt{f(x_1)})(y - \sqrt{f(x_2)})(y + \sqrt{f(x_2)})} \end{aligned} \quad (13)$$

as  $m$  tends to  $\infty$ .

This appears as the period integration of a regular singular connection

$$\begin{aligned} \nabla_{(m)} &= \left(d + \frac{(m+1)dy}{y+m}\right) \otimes \nabla' \\ &= d + I \frac{(m+1)dy}{y+m} + I \varpi \\ &\quad + \begin{pmatrix} \frac{1}{9}(\lambda+1)(2\lambda^2 - 3\lambda + 2) + y^2 & \frac{2}{9}\lambda(\lambda^2 + 1) - \frac{2}{3}(\lambda+1)y^2 \\ -\frac{2}{9}(\lambda^2 - \lambda + 1) & -\frac{2}{9}\lambda(\lambda+1) + 2y^2 \end{pmatrix} \omega. \end{aligned} \quad (14)$$

Note that  $\frac{(m+1)dy}{y+m}$  was chosen to approximate  $dy$ . For each  $m$ , calculating the period determinant of  $\nabla_{(m)}$ , we approximate that of  $\nabla_1$ . We obtain the Tame symbols of  $\nabla_{(m)}$  as follows:

$$\begin{aligned} (\nabla_{(m)}, y - \sqrt{f(x_1)})_{\gamma_1} &= (2\sqrt{f(x_1)})^{5/2} (f(x_1) - f(x_2))^{5/2} (\sqrt{f(x_1)} + m)^{2(m+1)}, \\ (\nabla_{(m)}, y + \sqrt{f(x_1)})_{\gamma_2} &= (-2\sqrt{f(x_1)})^{5/2} (f(x_1) - f(x_2))^{5/2} (-\sqrt{f(x_1)} + m)^{2(m+1)}, \\ (\nabla_{(m)}, y - \sqrt{f(x_2)})_{\gamma_3} &= (2\sqrt{f(x_2)})^{5/2} (f(x_2) - f(x_1))^{5/2} (\sqrt{f(x_2)} + m)^{2(m+1)}, \\ (\nabla_{(m)}, y + \sqrt{f(x_2)})_{\gamma_4} &= (-2\sqrt{f(x_2)})^{5/2} (f(x_2) - f(x_1))^{5/2} (-\sqrt{f(x_2)} + m)^{2(m+1)}, \\ (\nabla_{(m)}, y+m)_{\gamma_{(-m)}} &= (m^2 - f(x_1))^{5/2} (m^2 - f(x_2))^{5/2} \quad \text{and} \\ \left(\nabla_{(m)}, \frac{1}{y}\right)_{\gamma_\infty} &= 1. \end{aligned} \quad (15)$$

The Gamma factors are

$$\Gamma_{\pm\sqrt{f(x_i)}}(\nabla_{(m)}) = \Gamma(1)\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2},$$

$$\Gamma_{-m}(\nabla_{(m)}) = \Gamma(m+1)^2$$

and

$$\Gamma_{\infty}(\nabla_{(m)}) = \Gamma\left(m+1 + \frac{14}{3}\right) \cdot \Gamma\left(m+1 + \frac{16}{3}\right).$$

With the above factors, we have the period determinant for the regular singular connection  $\nabla_{(m)}$  with respect to the basis chosen below.

$$\begin{aligned} C_{j,b}^{(m)} &:= I_j^{(m)} \otimes \text{Sol}(\nabla_{(m)}^*)(e_b^*), \\ \omega_{1a}^{(m)} &:= \left( \frac{1}{y + \sqrt{f(x_1)}} - \frac{1}{y - \sqrt{f(x_1)}} \right) dy \otimes e_a, \\ \omega_{2a}^{(m)} &:= \left( \frac{1}{y - \sqrt{f(x_1)}} - \frac{1}{y + \sqrt{f(x_2)}} \right) dy \otimes e_a, \\ \omega_{3a}^{(m)} &:= \left( \frac{1}{y + \sqrt{f(x_2)}} - \frac{1}{y - \sqrt{f(x_2)}} \right) dy \otimes e_a, \\ \omega_{4a}^{(m)} &:= \left( \frac{1}{y - \sqrt{f(x_2)}} - \frac{1}{y + m} \right) dy \otimes e_a. \end{aligned} \quad (16)$$

The period determinant of  $\nabla_{(m)}$  with respect to  $\omega_{i,a}^{(m)}$  and  $C_{j,b}^{(m)}$  obtained by the application of Theorem 5 as follows:

$$\begin{aligned} D_{(m)} &= \det((C_{j,b}^{(m)}, \omega_{i,a}^{(m)})) \\ &= 2^6 \pi^2 f(x_1)^{5/2} f(x_2)^{5/2} (f(x_1) - f(x_2)) \\ &\quad \times (m^2 - f(x_1))^{2(m+1)+5/2} (m^2 - f(x_2))^{2(m+1)+5/2} \\ &\quad \times \frac{\Gamma(m+1)^2}{\Gamma((m+1) + \frac{14}{3}) \Gamma((m+1) + \frac{16}{3})}. \end{aligned} \quad (17)$$

The above value will not converge, but using Proposition 6 we obtain a convergent sequence of period determinants

$$P_{(m)} := \det(P_{(m),i,j,a,b}) = \left( \frac{1}{m} \right)^{8m} \frac{D_{(m)}}{\Delta_{(m)}^2} \quad (18)$$

where  $\Delta_{(m)}$  is the Vandermonde determinant for  $\pm\sqrt{f(x_1)}$ ,  $\pm\sqrt{f(x_2)}$ ,  $-m$  and  $\Delta_{(m)}$  is equal to

$$-2^2 \sqrt{f(x_1)} \sqrt{f(x_2)} (f(x_2) - f(x_1))^2 (m^2 - f(x_1)) (m^2 - f(x_2)).$$

Together with the above, the period determinant of  $\nabla_{(m)}$  with respect to  $\eta_{i,a}^{(m)}$  and  $C_{j,b}^{(m)}$  is

$$\begin{aligned} P_{(m)} &= \det(P_{(m),i,j,a,b}) = \det((\eta_{i,a}^{(m)}, C_{j,b}^{(m)})) = \left(\frac{1}{m}\right)^{8m} \frac{D_{(m)}}{\Delta_{(m)}^2} \\ &= \frac{2^2 \pi^2 f(x_1)^{3/2} f(x_2)^{3/2}}{(f(x_1) - f(x_2))^3} \\ &\quad \times \left(\frac{1}{m}\right)^{8m} (m^2 - f(x_1))^{2m} (m^2 - f(x_2))^{2m} \\ &\quad \times m^{-10} (m^2 - f(x_1))^{5/2} (m^2 - f(x_2))^{5/2} \\ &\quad \times m^{10} \frac{\Gamma(m+1)^2}{\Gamma((m+1) + \frac{14}{3}) \Gamma((m+1) + \frac{16}{3})}, \end{aligned} \quad (19)$$

which converges to  $P$  the period determinant of  $\nabla_1$ . Each term in the above converges as follows:

$$\begin{aligned} \left(\frac{1}{m}\right)^{8m} (m^2 - f(x_1))^{2m} (m^2 - f(x_2))^{2m} &\rightarrow 1, \\ m^{-10} (m^2 - f(x_1))^{5/2} (m^2 - f(x_2))^{5/2} &\rightarrow 1, \\ m^{10} \frac{\Gamma(m+1)^2}{\Gamma((m+1) + \frac{14}{3}) \Gamma((m+1) + \frac{16}{3})} &\rightarrow 1. \end{aligned} \quad (20)$$

Therefore  $P$  is

$$\frac{2^2 \pi^2 f(x_1)^{3/2} f(x_2)^{3/2}}{(f(x_1) - f(x_2))^3}.$$

Note the above value is found in  $\mathbb{C}^*$ .

The period determinant of  $\pi_*(\nabla)_D = \pi_* \pi^*(d + dy + \varpi)$  is then,

$$\begin{aligned} &(\text{Period of } (\mathcal{O}_{\mathbb{A}^1}, d + dy + \varpi)) \times (\text{Period of } \nabla_1) \\ &= \frac{2^4 \pi^2 f(x_1)^2 f(x_2)^2}{f(x_1) - f(x_2)}. \end{aligned} \quad (21)$$

The above period determinant is obtained by taking the rank 2 part cohomology basis  $\eta_{i,a}$  ( $a = 1, 2$ ) as in (11) and

$$\eta_{i,0} := \frac{y^{i-1} dy}{(y^2 - f(x_1))(y^2 - f(x_2))}$$

the basis of  $H_{\text{dR}}^1(\mathbb{A}^1, d + dy + \varpi)$ . Recovering  $e_1, e_2$  back to  $-\frac{\lambda+1}{3} + x, -\frac{\lambda^2+1}{3} + x^2$ , one can easily find that the period determinant is equal to the that calculated with the following basis of  $H_{\text{dR}}^1(\mathbb{A}^1, \pi_*(\nabla_D))$ :

$$\bar{\eta}_{i,j} := \frac{x^{i-1}y^{j-1}dy}{(y^2 - f(x_1))(y^2 - f(x_2))} \quad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4. \quad (22)$$

This is due to the fact that the base change of  $\langle 1, e_1, e_2 \rangle$  to  $\langle 1, x, x^2 \rangle$  has determinant 1.

**Proposition 8.** Take  $\bar{\eta}_{i,j}$  as basis of  $H_{\text{dR}}^1(\mathbb{A}^1, \pi_*(\nabla_D))$ . For  $H_1^{\text{irreg}}(\mathbb{A}^1, D, \pi_*(\nabla_D)^*)$ , take  $I_j \otimes \exp(y)(y^2 - f(x_1))(y^2 - f(x_2)), C_{j,b}$  as a basis. Then the period determinant of the connection  $(\mathbb{A}^1, \pi_*(\nabla_D))$  w.r.t. this basis is

$$\frac{2^4 \pi^2 f(x_1)^2 f(x_2)^2}{f(x_1) - f(x_2)} \in \mathbb{C}^*.$$

## 6. Comparison

Recall that  $U$  is the affine Legendre elliptic curve defined by the equation  $y^2 = x(x-1)(x-\lambda)$  and  $\nabla = d + dy$ . Let  $D$  be as in the previous section and  $\Sigma := \pi^{-1}D$  in  $U$ . Then using the projection formula,  $\pi_*(\nabla)_D = d_D \otimes \pi_*\nabla$ , where  $d_D$  and  $\nabla_D$  are defined as before. In the same manner, we denote by  $\nabla_\Sigma$ , the twisted connection  $d_\Sigma \otimes \nabla$  on  $\mathcal{O}_U(-\Sigma)$ .

Then  $\pi$  induces a canonical isomorphism of de Rham cohomologies:

$$\pi^*: H_{\text{dR}}^1(\mathbb{A}^1, \nabla_D) \rightarrow H_{\text{dR}}^1(U, \nabla_\Sigma).$$

If  $\omega = \pi^*\eta$  is a de Rham form in  $H_{\text{dR}}^1(U, \nabla_\Sigma)$  for a  $\eta$  in  $H_{\text{dR}}^1(\mathbb{A}^1, \nabla_D)$ , then the functoriality of the pairing implies  $\langle \gamma, \omega \rangle = \langle \gamma, \pi^*\eta \rangle = \langle \pi_*\gamma, \eta \rangle$ . Hence the period of  $\pi_*(\nabla)_D$  that we calculated previously is the same as that of  $\nabla_\Sigma$  on  $U$ .

The following short exact sequence of de Rham complexes

$$\begin{array}{ccccc} \mathcal{O}_U(-\Sigma) & \longrightarrow & \mathcal{O}_U & \longrightarrow & \mathcal{O}_\Sigma \\ \downarrow \nabla_\Sigma & & \downarrow \nabla & & \downarrow \\ \Omega_U(-\Sigma)(\log \Sigma) & \xlongequal{\quad} & \Omega_U & \longrightarrow & 0 \end{array} \quad (23)$$

yields

$$H_{\text{dR}}^0(U, \nabla) \rightarrow H^0(U, \mathcal{O}_\Sigma) \xrightarrow{\nabla} H_{\text{dR}}^1(U, \nabla_\Sigma) \rightarrow H_{\text{dR}}^1(U, \nabla) \rightarrow H^1(U, \mathcal{O}_\Sigma). \quad (24)$$

Since  $\nabla$  admits no (single-valued) solution on  $U$ ,  $H_{\text{dR}}^0(U, \nabla) = 0$  and  $H^1(U, \mathcal{O}_\Sigma)$  vanishes by dimension reason,  $H_{\text{dR}}^1(U, \nabla_\Sigma)$  is an extension of  $H_{\text{dR}}^1(U, \nabla)$  by  $H^0(U, \mathcal{O}_\Sigma)$ .



In the homology side,  $H_1^{\text{irreg}}(U, \Sigma, \nabla^*)$  is isomorphic to  $H_1^{\text{irreg}}(\mathbb{A}^1, D, \pi_*(\nabla^*))$  via  $\pi_*$ . The relative cohomology appears in the short exact sequence of the singular complexes valued in  $\nabla^*$ :

$$0 \rightarrow \mathcal{C}_*^{\text{irreg}}(\Sigma, \nabla^*) \rightarrow \mathcal{C}_*^{\text{irreg}}(U, \nabla^*) \rightarrow \mathcal{C}_*^{\text{irreg}}(U, \Sigma, \nabla^*) \rightarrow 0, \quad (25)$$

which also yields a long exact sequence of homologies:

$$\begin{aligned} H_1^{\text{irreg}}(\Sigma, \nabla^*) &\rightarrow H_1^{\text{irreg}}(U, \nabla^*) \rightarrow H_1^{\text{irreg}}(U, \Sigma, \nabla^*) \\ &\rightarrow H_0^{\text{irreg}}(\Sigma, \nabla^*) \rightarrow H_0^{\text{irreg}}(U, \nabla^*). \end{aligned} \quad (26)$$

In the above sequence, as in cohomology side, we have vanishing  $H_1^{\text{irreg}}(\Sigma, \nabla^*)$  and  $H_0^{\text{irreg}}(U, \nabla^*)$ , respectively, by dimension reason and by the duality of de Rham cohomology and homology for irregular connections.

**Theorem 9.** *Let  $\gamma \otimes \text{Sol}(\nabla^*)$  be a cycle in  $H_1^{\text{irreg}}(U, \nabla^*)$ , thus a cycle in  $H_1^{\text{irreg}}(U, \Sigma, \nabla^*)$ . Suppose  $\omega = \nabla f$  for a function  $f$  in  $H^0(U, \mathcal{O}_\Sigma)$ , then the pairing of  $\gamma \otimes \text{Sol}(\nabla^*)$  with  $\nabla f$  is 0.*

**Proof.** The pairing is the integration  $\langle \gamma \otimes \text{Sol}(\nabla^*), \nabla \rangle = \int_\gamma \langle \text{Sol}(\nabla^*), \nabla f \rangle$ . This is equal to

$$\begin{aligned} &\int_\gamma (d\langle \text{Sol}(\nabla^*), f \rangle - \langle \nabla^*(\text{Sol}(\nabla^*)), f \rangle) \\ &= \int_\gamma d\langle \text{Sol}(\nabla^*), f \rangle = \int_{\partial\gamma} \langle \text{Sol}(\nabla^*), f \rangle \end{aligned} \quad (27)$$

by Stokes' theorem.

For a closed cycle  $\gamma$ ,  $\partial\gamma = 0$ . In this case we have nothing to prove. Otherwise, a tubular neighborhood of  $\partial\gamma$  in  $\gamma$  lies in the rapid decay sector of  $\nabla^*$  around  $\infty$ , thus the integration yields 0.  $\square$

Using the previous theorem, we conclude the following:

**Corollary 10.** *The period determinant of  $\nabla_\Sigma$  is*

$$\text{per}(U, \Sigma, \nabla_\Sigma) = \text{per}(U, \nabla) \times \text{per}(\Sigma, \nabla)$$

in  $\mathbb{C}^*$ .

To calculate exact period determinant in  $\mathbb{C}^*$ , we have to fix an appropriate basis for  $H_{\text{dR}}^1(U, \nabla)$  and  $H^0(U, \mathcal{O}_\Sigma)$ . Otherwise, the period determinant would be obtained modulo  $k^*$ . Recall we have fixed

$$\{xy^{i-1}dy \mid i = 1, 2, 3, 4\}$$

the basis of  $H_{\text{dR}}^1(U, \nabla)$  for the approximation in the previous section,

$$\{x^{i-1}y^{j-1}dy \mid i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4\}$$

is chosen for the basis of  $H_{\text{dR}}^1(U, \nabla_\Sigma)$ .

$\Sigma$  is given by the principal ideal  $I := (\frac{(y^2-f(x_1))(y^2-f(x_2))}{(x-x_1)(x-x_2)})$  in  $R := k[x, y]/(y^2 - f(x))$ . It follows  $H^0(U, \mathcal{O}_\Sigma) = R/I$ . So we have a basis:  $\{1, y, y^2, y^3, x, xy, xy^2, xy^3\}$  of  $H^0(U, \mathcal{O}_\Sigma)$ , whereas  $H_0^{\text{irreg}}(\Sigma, \nabla^*)$  has a basis  $\{p \otimes \exp y \mid p \in \Sigma\}$ . But we should be careful as the image of this basis via  $\nabla$  does not cover properly the complement of  $\{xy^{j-1}dy \mid i = 1, 2, 3\}$  in  $\{x^{i-1}y^{j-1}dy \mid i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4\}$  (i.e.,  $\{xy^{j-1}dy \mid i = 1, 2, 3\} \cup \{\nabla(x^{i-1}y^{j-1}) \mid i = 1, 2 \text{ and } j = 1, 2, 3, 4\}$  and  $\{x^{i-1}y^{j-1}dy \mid i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4\}$  are not related by a matrix of determinant 1).

Instead we take the image of  $\{y^{i-1} \mid i = 1, 2, 3, 4\} \cup \{(\frac{1}{2}x^2 - x_1x)(y^2 - f(x_2))y^{i-1}, (\frac{1}{2}x^2 - x_2x)(y^2 - f(x_1))y^{i-1} \mid i = 1, 2\}$  by  $\nabla$  as the complementary basis of  $H_{\text{dR}}^1(U, \nabla)$  in  $H^1(U, \nabla_\Sigma)$ .

The image of this complementary basis is

$$\nabla(y^i) = iy^{i-1}dy + y^i dy, \quad (28)$$

and for  $i = 1, 2$ . This basis  $\{\nabla(y^i)\}$  is related to  $\{y^i dy\}$  by a matrix of determinant 1. Thus we can replace  $y^i dy$  with  $\nabla(y^i)$  without changing the period determinant.

The rests are as follows:

$$\begin{aligned} & \nabla\left(\left(\frac{1}{2}x^2 - x_1x\right)(y^2 - f(x_2))\right) \\ &= (x - x_1)(y^2 - f(x_2))dx + \left(\frac{1}{2}x^2 - x_1x\right)2y dy \\ &+ \left(\frac{1}{2}x^2 - x_1x\right)(y^2 - f(x_2))dy \end{aligned} \quad (29)$$

where  $x_1, x_3$  are the two roots of

$$\begin{aligned} \frac{f(x) - f(x_1)}{x - x_1} &= (x - x_1)(x - x_3) \\ &= x^2 + (x_1 - (\lambda + 1))x + x_1^2 - (\lambda + 1)x_1 + \lambda. \end{aligned} \quad (30)$$

Thus, we obtain  $\nabla((\frac{1}{2}x^2 - x_1x)(y^2 - f(x_2)))$  modulo  $y^i dy$  and  $xy^i dy$ :

$$\nabla\left(\left(\frac{1}{2}x^2 - x_1x\right)(y^2 - f(x_2))\right) \sim \frac{5}{3}x^2 dy + \frac{1}{2}x^2 y^2 dy - \frac{1}{2}x^2 f(x_2) dy. \quad (31)$$

By the same way,

$$\begin{aligned} \nabla\left(\left(\frac{1}{2}x^2 - x_2x\right)(y^2 - f(x_1))\right) &\sim \frac{5}{3}x^2 dy + \frac{1}{2}x^2 y^2 dy - \frac{1}{2}x^2 f(x_1) dy, \\ \nabla\left(\left(\frac{1}{2}x^2 - x_1x\right)(y^2 - f(x_2))y\right) &\sim \frac{13}{6}x^2 y^2 dy - \frac{f(x_2)}{2}x^2 dy \\ &\quad + \frac{1}{2}x^2 y^2 dy - \frac{f(x_2)}{2}x^2 y dy, \\ \nabla\left(\left(\frac{1}{2}x^2 - x_2x\right)(y^2 - f(x_1))y\right) &\sim \frac{13}{6}x^2 y^2 dy - \frac{f(x_1)}{2}x^2 dy \\ &\quad + \frac{1}{2}x^2 y^2 dy - \frac{f(x_1)}{2}x^2 y dy. \end{aligned} \quad (32)$$

This change of basis has determinant

$$D = \frac{(f(x_1) - f(x_2))^2}{16}. \quad (D)$$

Note that the above change of basis is only valid inside  $H_{\text{dR}}^1(U, \nabla_\Sigma)$  as we have to identify the two vector spaces modulo  $y^i dy$  and  $xy^i dy$ .

Now we have to calculate the determinant of the period matrix arising from the pairing of  $\text{Im}(H^0(U, \mathcal{O}_\Sigma) \xrightarrow{\nabla} H_{\text{dR}}^1(U, \nabla_\Sigma))$  and  $H_0^{\text{irreg}}(\Sigma, \nabla^*)$ .

Using Theorem 9, as the cohomology basis are  $\nabla$ -exact, the above periods are nothing but the evaluation of the primitive functions w.r.t.  $\nabla$  at the points of  $\Sigma$ . To take full benefit of symmetry of matrix, we reorder the basis to obtain the following period matrix:

$$\begin{pmatrix} L_1 N & L_1 M \\ L_2 N & -L_2 M \end{pmatrix}$$

where  $L_1, L_2, M, N$  are defined as follows:

- $L_1 = (\exp(\sqrt{f(x_i)})\delta_{ij})_{i=1,2,3,4}$ ,  $L_2 = (\exp(-\sqrt{f(x_i)})\delta_{ij})_{i=1,2,3,4}$ .
- $L_1 N$  is the period matrix for  $(x_i, \sqrt{f(x_i)})_{i=1,2,3,4} \otimes \exp y$  and  $\{1, y^2, (\frac{1}{2}x^2 - x_1x) \times (y^2 - f(x_2)), (\frac{1}{2}x^2 - x_2x)(y^2 - f(x_1))\}$  so that

$$N = \begin{pmatrix} 1 & f(x_1) & -\frac{1}{2}x_1^2(f(x_1) - f(x_2)) & 0 \\ 1 & f(x_2) & 0 & -\frac{1}{2}x_2^2(f(x_2) - f(x_1)) \\ 1 & f(x_1) & (\frac{1}{2}x_3^2 - x_1x_3)(f(x_1) - f(x_2)) & 0 \\ 1 & f(x_2) & 0 & (\frac{1}{2}x_4^2 - x_2x_4)(f(x_2) - f(x_1)) \end{pmatrix}.$$

- $L_1 M$  is the period matrix for  $(x_i, \sqrt{f(x_i)})_{i=1,2,3,4} \otimes \exp y$  and  $\{y, y^3, \frac{1}{2}(x^2 - x_1) \times (y^2 - f(x_2))y, \frac{1}{2}(x^2 - x_2x)(y^2 - f(x_1))y\}$  so that

$$M = \begin{pmatrix} \sqrt{f(x_1)} & & & 0 \\ & \sqrt{f(x_2)} & & \\ & & \sqrt{f(x_1)} & \\ 0 & & & \sqrt{f(x_2)} \end{pmatrix} N.$$

Therefore using elementary operations which does not change the determinant, we have

$$\begin{aligned} \det \begin{pmatrix} L_1 N & L_1 M \\ L_2 N & -L_2 M \end{pmatrix} &= \det \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \det \begin{pmatrix} N & M \\ N & -M \end{pmatrix} \\ &= \det \begin{pmatrix} N & M \\ 2N & 0 \end{pmatrix} = 2^4 \det(N) \det(M) \\ &= 2^4 f(x_1)^2 f(x_2)^2 \det(N)^2. \end{aligned} \quad (33)$$

Now it remains to calculate  $\det(N)$ :

$$\begin{aligned} \det(N) &= \begin{vmatrix} 1 & f(x_1) & -\frac{1}{2}x_1^2(f(x_1) - f(x_2)) & 0 \\ 1 & f(x_2) & 0 & -\frac{1}{2}x_2^2(f(x_2) - f(x_1)) \\ 1 & f(x_1) & (\frac{1}{2}x_3^2 - x_1x_3)(f(x_1) - f(x_2)) & 0 \\ 1 & f(x_2) & 0 & (\frac{1}{2}x_4^2 - x_2x_4)(f(x_2) - f(x_1)) \end{vmatrix} \\ &= \begin{vmatrix} 1 & f(x_1) & * & * \\ 1 & f(x_2) & * & * \\ 0 & 0 & \frac{1}{2}(x_1^2 - 2x_1x_3 + x_3^2)(f(x_1) - f(x_2)) & 0 \\ 0 & 0 & 0 & \frac{1}{2}(x_4^2 - x_2x_4 + x_2^2)(f(x_2) - f(x_1)) \end{vmatrix} \\ &= -2^{-2} \begin{vmatrix} 1 & f(x_1) \\ 1 & f(x_2) \end{vmatrix} (x_1 - x_3)^2 (x_2 - x_4)^2 (f(x_2) - f(x_1))^2 \\ &= 2^{-2} (x_1 - x_3)^2 (x_2 - x_4)^2 (f(x_1) - f(x_2))^3. \end{aligned} \quad (34)$$

Putting the value into the previous calculation, we obtain the period determinant w.r.t. the basis  $\nabla(y^i)$  and  $\nabla((\frac{1}{2}x^2 - x_i)(y^2 - f(x_j))y^a)$ :

$$\det \begin{pmatrix} L_1 N & L_1 M \\ L_2 N & -L_2 M \end{pmatrix} = 2^{-2} f(x_1)^2 f(x_2)^2 (x_1 - x_3)^4 (x_2 - x_4)^4 (f(x_1) - f(x_2))^6.$$

Therefore the exact determinant inside  $H_{\text{dR}}^1(U, \nabla)$  is the above divided by (D):

$$\text{per}(\Sigma, \nabla) = 2^{-2} 3^{-6} f(x_1)^2 f(x_2)^2 (x_1 - x_3)^4 (x_2 - x_4)^4 (f(x_1) - f(x_2))^4.$$

After all, we obtain the period of  $\nabla = d + dy$  over  $U$ :

$$\begin{aligned}\operatorname{per}(U, \nabla) &= \frac{\operatorname{per}(U, \nabla_{\Sigma})}{\operatorname{per}(\Sigma, \nabla)} \\ &= \frac{2^6 3^6 \pi^2}{(x_1 - x_3)^4 (x_2 - x_4)^4 (f(x_1) - f(x_2))^5}.\end{aligned}\quad (35)$$

To see the value explicitly, we need the values of  $f(x_1) - f(x_2)$  and  $(x_1 - x_3)^2 \times (x_2 - x_4)^2$ .

$$\begin{aligned}(f(x_1) - f(x_2))^2 &= (f(x_1) + f(x_2))^2 - 4f(x_1)f(x_2) \\ &= \frac{2^4}{3^6}(\lambda^2 - \lambda + 1)^3.\end{aligned}\quad (36)$$

It follows  $f(x_1) - f(x_2) = \frac{2^2}{3^3}(\lambda^2 - \lambda + 1)\sqrt{\lambda^2 - \lambda + 1}$ , for a suitable choice of the branch of the square root.

Recall that  $x_1, x_3$  are the two roots of

$$\frac{f(x) - f(x_1)}{x - x_1} = x^2 + (x_1 - (\lambda + 1))x + x_1^2 - (\lambda + 1)x_1 + \lambda.$$

Thus from the coefficients of this quadratic equation,

$$(x_1 - x_3)^2 = \frac{-3x_1^2 + 2(\lambda + 1)x_1 + (\lambda + 1)^2}{4}.$$

By the same way, we obtain

$$(x_2 - x_4)^2 = \frac{-3x_2^2 + 2(\lambda + 1)x_2 + (\lambda + 1)^2}{4}$$

and thus we deduce

$$(x_1 - x_3)^2 (x_2 - x_4)^2 = 2^{-4}(\lambda^2 - \lambda + 1)^2. \quad (37)$$

Finally, we have the period explicitly:

**Theorem 11.** *Let  $\nabla = d + dy$  a connection over the affine Legendre elliptic curve  $y^2 = x(x - 1)(x - \lambda)$ . If we fix the basis  $\{x dy, xy dy, xy^2 dy, xy^3 dy\}$  for  $H_{\text{dR}}^1(U, \nabla)$ , the period determinant for a suitable choice of basis for  $H_1^{\text{irreg}}(U, \nabla^*)$  where the topological cycles have value  $\exp y$  for  $\lambda \neq 0, 1, (1 \pm \sqrt{-3})/2$  is*

$$\frac{2^4 3^{21} \pi^2}{(\lambda^2 - \lambda + 1)^{11} \sqrt{\lambda^2 - \lambda + 1}}$$

in  $\mathbb{C}^*$ .

**Remark 12.** The above period determinant is well defined in  $\mathbb{C}^*/\mathbb{Q}^*$  for the choice of basis which is valued at  $\exp y$ .

## 7. Exceptional case: $\lambda^2 - \lambda + 1 = 0$

Finally, we handle the case  $\lambda^2 - \lambda + 1 = 0$ . Recall that  $x_1 = \frac{\lambda+1}{3}$  is the double root of  $f'(x)$  as well as the triple root of  $f(x) - f(x_1)$ .

In this case, using the projection formula again, we find  $\pi_*(\nabla) \simeq (\mathcal{O}_{\mathbb{A}^1}, d + dy) \oplus (\ker \text{Tr}, (d + dy) \otimes \nabla')$ , where  $\nabla'$  is, as before, the connection on the rank 2 part of  $\pi_*(\mathcal{O}_U, d)$ :

$$\nabla' = d + \begin{pmatrix} 1 & -\frac{2}{3}(\lambda + 1) \\ 0 & 2 \end{pmatrix} \omega'$$

with  $\omega' = \frac{2y dy}{3(y^2 - f(x_1))}$ .

$(d + dy) \otimes \nabla'$  has regular singularities at  $D = \{\pm\sqrt{f(x_1)}\}$ . At each point of  $D$ , the residue of  $\nabla'$  has two eigenvalues  $1/3, 2/3$ . For this connection, one can directly apply Terasoma's product formula.

Since  $H_{\text{dR}}^1(\mathbb{A}^1, d + dy) = 0$ ,

$$H_{\text{dR}}^1(U, \nabla) \simeq H_{\text{dR}}^1(\mathbb{A}^1, d + dy) \oplus H_{\text{dR}}^1(\mathbb{A}^1, (d + dy) \otimes \nabla'). \quad (38)$$

Therefore  $(\mathcal{O}_{\mathbb{A}^1}^2, (d + dy) \otimes \nabla')$  yields the same period determinant as  $(\mathcal{O}_U, \nabla)$ :

$$\text{per}(U, \nabla) = \text{per}(U, (d + dy) \otimes \nabla') = \text{per}(U, \pi_*(\nabla)).$$

Let  $\Sigma := \pi^{-1}D = \{x_1, \pm\sqrt{f(x_1)}\} \subset U$ .

Using this identification the period matrix will be approximated as in the previous section.

For the approximation, we have to find the cycles generating  $H_1^{\text{irreg}}(\mathbb{A}^1, ((d + dy) \otimes \nabla')^*)$ . Let  $\gamma_1, \gamma_2$  be a fixed path from 0 to  $-\sqrt{f(x_1)}$  and to  $\sqrt{f(x_1)}$  in  $\mathbb{A}^1$ , respectively.  $\gamma_m$  is a path from 0 to  $-m$ . Define

$$I_1^{(m)} := \gamma_2 - \gamma_1 \quad \text{and} \quad I_2^{(m)} := \gamma_m - \gamma_2.$$

Note that  $I_1^{(m)}$  is independent of  $m$  and  $I_2^{(m)}$  goes to  $\infty$  along the rapid decay sector of  $\nabla^*$ . Let

$$I_1 = I_1^{(m)} \quad \text{and} \quad I_2 = \lim_{m \rightarrow \infty} I_2^{(m)}.$$

Then  $I_i \otimes \text{Sol}(\nabla^*)(e_p^*)$  for  $i, p = 1, 2$  make a basis for  $H_1^{\text{irreg}}(\mathbb{A}^1, ((d + dy) \otimes \nabla')^*)$ .

In the other hand,  $\eta_i(e_q) := \frac{y^{i-1}}{y^2-f(x_1)} dy \otimes e_q$  for  $i = 1, 2$ , which is a basis for  $H_{\text{dR}}^1(\mathbb{A}^1, (d + dy) \otimes \nabla')$ .

Let  $\eta_i$  be  $\frac{y^{i-1} dy}{y^2-f(x_1)}$ . We introduce a new connection  $\nabla^{(m)} := (d + \frac{m+1}{m} dy) \otimes \nabla'$  which approximates the rank 2 part of  $\pi_*(\nabla)$ . Then the  $(i, j, p, q)$  entry of the period matrix is

$$\begin{aligned} & \langle I_j \otimes \exp y \text{Sol}(\nabla'^*) (e_p^*), \eta_i(e_q) \rangle \\ &= \int_{I_j} \langle \text{Sol}(\nabla'^*) (e_p^*), e_q \rangle \exp y \cdot \eta_i \\ &= \lim_{m \rightarrow \infty} \frac{1}{m^m} \int_{I_j^{(m)}} \langle \text{Sol}(\nabla'^*) (e_p^*), e_q \rangle \frac{(y+m)^m y^{i-1}}{y^2-f(x_1)} dy \\ &= \lim_{m \rightarrow \infty} \frac{1}{m^m} \langle I_j^{(m)} \otimes \text{Sol}(\nabla^{(m)}) (e_q^*), \eta_i^{(m)}(e_p) \rangle \end{aligned} \quad (39)$$

where  $\eta_i^{(m)}(e_p)$  is the de Rham form  $\frac{y^{i-1} dy}{(y^2-f(x_1))(y+m)} \otimes e_p$ .

We denote the determinant of the above period matrix by  $P_{(m)}$ .

Applying the product formula, we obtain

$$\begin{aligned} P_{(m)} &= \frac{1}{m^{4m} \Delta_{(m)}^2} (\nabla^{(m)}, y - \sqrt{f(x_1)})_{\gamma_1} \cdot (\nabla^{(m)}, y + \sqrt{f(x_1)})_{\gamma_2} \\ &\quad \times (\nabla^{(m)}, y + m)_{\gamma_m} \cdot \left( \nabla^{(m)}, \frac{1}{y} \right)_{\infty}^{-1} \\ &\quad \times \Gamma(\nabla^{(m)})_{-\sqrt{f(x_1)}} \cdot \Gamma(\nabla^{(m)})_{\sqrt{f(x_1)}} \cdot \Gamma(\nabla^{(m)})_{-m} \cdot \Gamma(\nabla^{(m)})_{\infty}^{-1}. \end{aligned} \quad (40)$$

The tame symbols are

$$(\nabla^{(m)}, y - \sqrt{f(x_1)})_{\gamma_1} = 2\sqrt{f(x_1)}(\sqrt{f(x_1)} + m)^{2(m+1)}, \quad (41)$$

$$(\nabla^{(m)}, y + \sqrt{f(x_1)})_{\gamma_2} = -2\sqrt{f(x_1)}(-\sqrt{f(x_1)} + m)^{2(m+1)}, \quad (42)$$

$$(\nabla^{(m)}, y + m)_{\gamma_m} = m^2 - f(x_1), \quad (43)$$

and

$$\left( \nabla^{(m)}, \frac{1}{y} \right)_{\infty} = 1. \quad (44)$$

And the Gamma factors are

$$\Gamma_{\pm\sqrt{f(x_1)}}(\nabla^{(m)}) = \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right), \quad (45)$$

$$\Gamma_{-m}(\nabla^{(m)}) = \Gamma(m+1)^2, \quad (46)$$

$$\Gamma_{\infty}(\nabla^{(m)}) = \Gamma\left(\frac{2}{3} + (m+1)\right) \cdot \Gamma\left(\frac{4}{3} + (m+1)\right). \quad (47)$$

The period determinant  $P_{(m)}$  is the product of the followings, which converges:

$$\Gamma\left(\frac{1}{3}\right)^2 \Gamma\left(\frac{2}{3}\right)^2, \quad (48)$$

$$\frac{(m^2 - f(x_1))^{2m}}{m^{4m}} \rightarrow 1, \quad (49)$$

$$\frac{m^2 - f(x_1)}{m^2} \rightarrow 1, \quad (50)$$

$$\frac{m^2 \Gamma(m+1)^2}{\Gamma(\frac{2}{3} + (m+1)) \Gamma(\frac{4}{3} + (m+1))} \rightarrow 1. \quad (51)$$

From the functional equation  $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$  satisfied by the Gamma function, we obtain

$$\Gamma\left(\frac{1}{3}\right)^2 \Gamma\left(\frac{2}{3}\right)^2 = \left(\frac{\pi}{\sin(\frac{\pi}{3})}\right)^2 = \frac{2^2 \pi^2}{3}.$$

Therefore,

$$\text{per}(U, \nabla) = \Gamma\left(\frac{1}{3}\right)^2 \Gamma\left(\frac{2}{3}\right)^2 = \frac{2^2 \pi^2}{3}.$$

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